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On some lattice theoretic properties in the lattice of subgroups of the symmetric groups

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Abstract: **In this paper we study about some lattice theoretic properties in the lattice of subgroups of S_n. The lattice theoretic properties verified are: distributivity, general disjointness condition, supersolvability, supermodularity, semi-supermodularity, pseudo-complementedness in $L(S_n)$.**

Keywords: **symmetric group, lattice of subgroups, lattice theoretic properties.**

1. INTRODUCTION

In this paper we examine some lattice theoretic properties in the subgroup lattices. We have given "The structure of the lattice of subgroups of the symmetric group S_5" in [12] and we have investigated there some basic properties satisfied by it. In this paper we study about $L(S_n)$ is distributive and pseudo complemented if and only if $n \leq 2$. We prove also that $L(S_n)$ is satisfy the general disjointness condition, supermodular, semi-supermodular if and only if $n \leq 3$. We prove also that every atom in $L(S_n)$ is non modular when $n = 4$ and $n = 5$. We also prove that $L(S_n)$ is supersolvable when $n = 2$ and $n = 3$ and $L(S_n)$ is not supersolvable when $n = 4$ and $n = 5$. For $n > 5$, we cannot decide the supersolvability and it remains an open problem.

2. PRELIMINARY

We recall some lattice theoretic definitions that will be used later.

Definition 2.1

A Lattice L is said to be ***distributive*** if $a \vee (b \wedge c) = [(a \vee b) \wedge (a \vee c)]$ for all $a, b, c \in L$.

Definition 2.2

An element $a \in P$ is called an ***atom***, if $a \succ 0$ and it is a dual atom, if $a \prec 1$.

Definition 2.3

A Lattice L is said to be ***supersolvable***, if it contains a maximal chain called an M-chain in which every element is modular. By a modular element m in a lattice L, we mean $x \vee (m \wedge y) = (x \vee m) \wedge y$ whenever $x \leq y$ in L.

Definition 2.4

A lattice L with 0 satisfies the ***general disjointness*** property (GD) if $x \wedge y = 0$ and $(x \vee y) \wedge z = 0$ implies that $x \wedge (y \vee z) = 0$, for $x, y \in L$.

Definition 2.5

A lattice L is said to be ***supermodular*** if it satisfies the following identity $(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a \vee [b \wedge c \wedge (a \vee d)] \vee [c \wedge d \wedge (a \vee b)] \vee [b \wedge d \wedge (a \vee c)]$ for all $a, b, c, d \in L$.

Definition 2.6

A lattice L is said to be ***semi-supermodular*** if it satisfies the following identity $(a \vee x_1) \wedge (a \vee x_2) \wedge (a \vee x_3) \wedge (a \vee x_4) = a \vee [x_1 \wedge x_2 \wedge (a \vee x_3) \wedge (a \vee x_4)] \vee [x_1 \wedge x_3 \wedge (a \vee x_2) \wedge (a \vee x_4)] \vee [x_1 \wedge x_4 \wedge (a \vee x_2) \wedge (a \vee x_3)] \vee [x_2 \wedge x_3 \wedge (a \vee x_1) \wedge (a \vee x_4)] \vee [x_2 \wedge x_4 \wedge (a \vee x_1) \wedge (a \vee x_3)] \vee [x_3 \wedge x_4 \wedge (a \vee x_1) \wedge (a \vee x_2)]$ for all a, x_1, x_2, x_3, x_4 in L.

Definition 2.7

Let L be a lattice with 0. An element a* ∈ L is called a ***pseudo complement*** of a ∈ L if $a \wedge a^* = 0$ and if for any x ∈ L with $x \wedge a = 0$ we have $x \leq a^*$. A lattice L with 0 is called pseudo complemented if every element of L has a pseudo complement.

We produce below the structure of the lattice of subgroups of the symmetric groups S_2, S_3, S_4 and S_5.

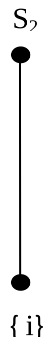

Fig. 2.1: Lattice Structure of L(S_2)

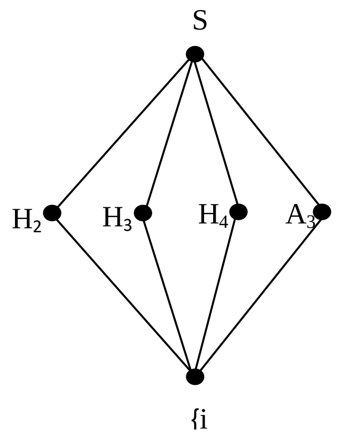

Fig. 2.2 : Lattice Structure of L(S_3)

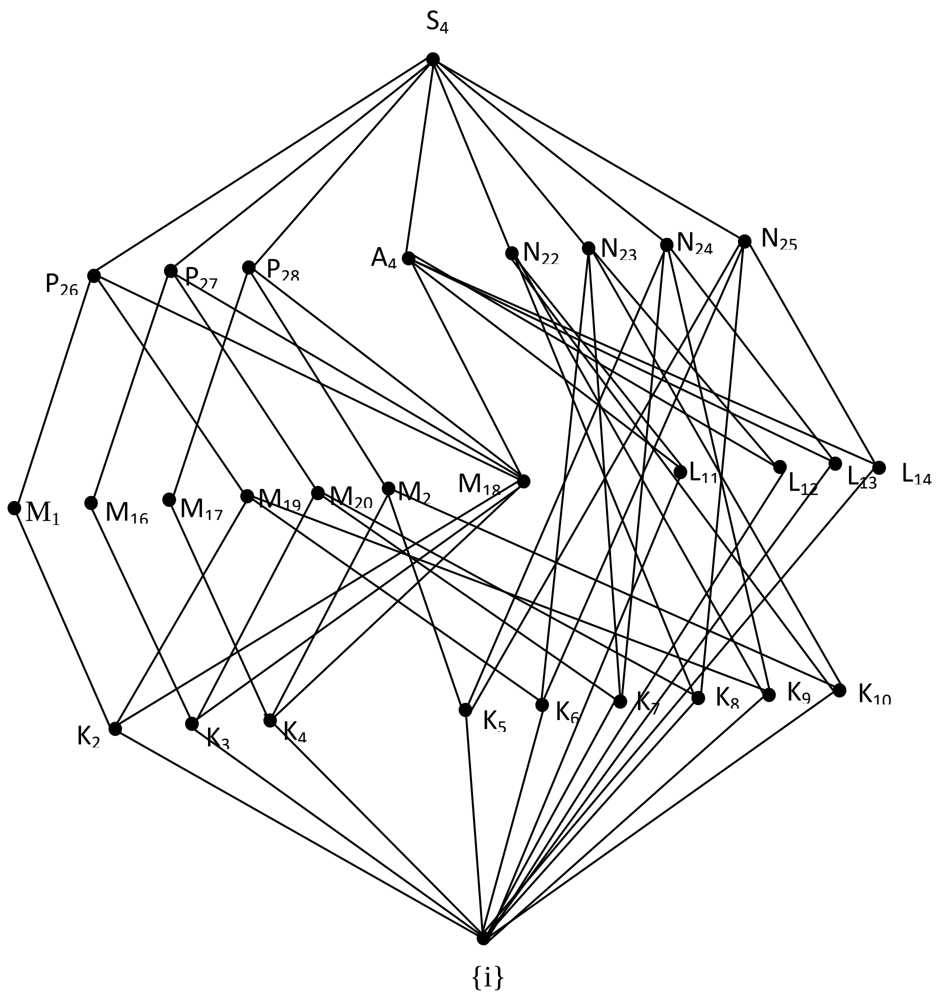

Fig. 2.3: Lattice Structure of L(S_4) [7]

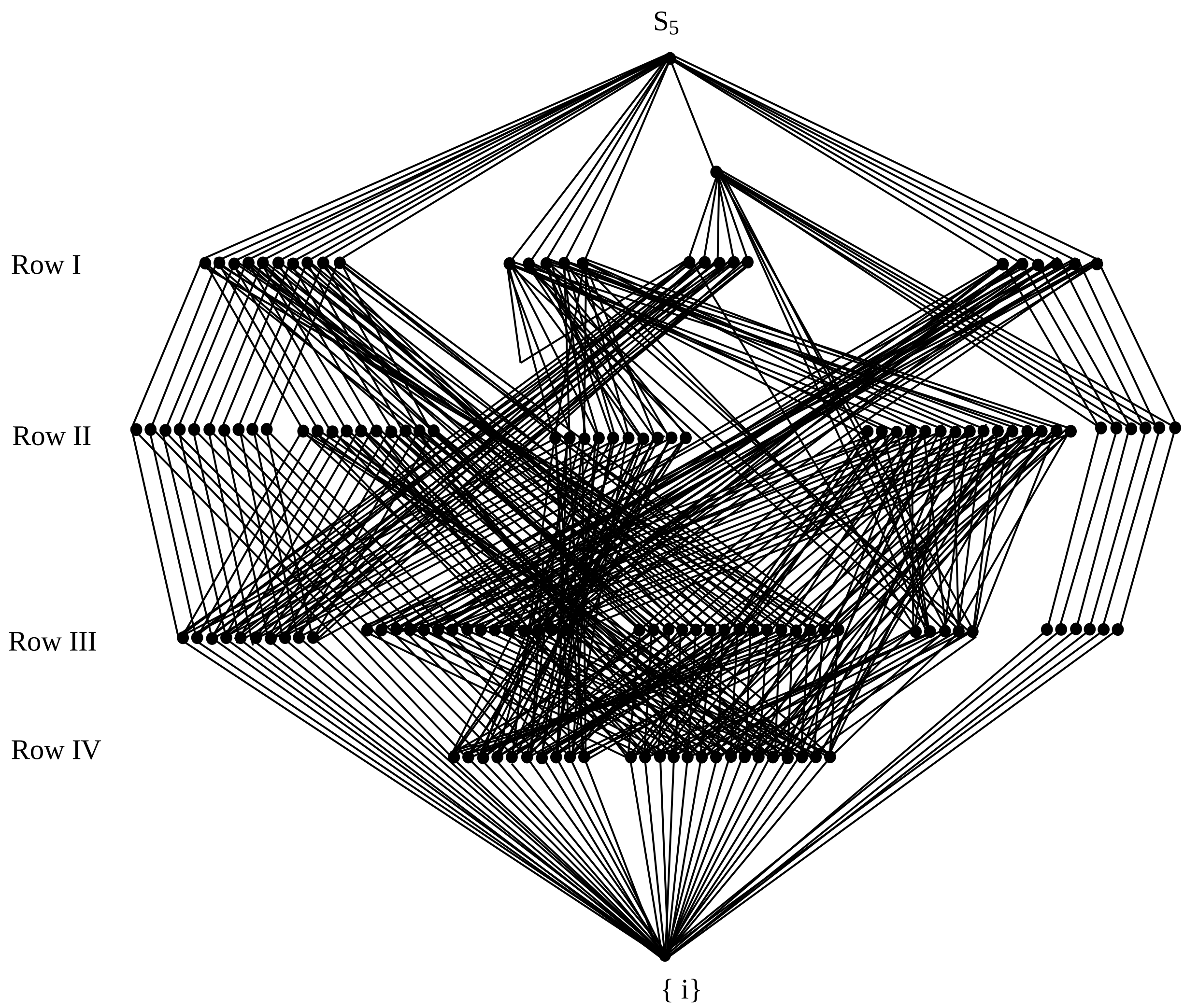

Fig. 2.4: Lattice Structure of $L(S_5)$ [12]

Row I : (Left to right): R_{129} to R_{138}, V_{150} to V_{154}, R_{139} to R_{143} and T_{144} to T_{149}

Row II : (Left to right): N_{78} to N_{87}, N_{98} to N_{107},N_{88} to N_{97}, Q_{108} to Q_{122} and P_{123} to P_{128}

Row III: (Left to right): K_{27} to K_{36}, L_{37} to L_{51}, L_{52} to L_{66}, L_{67} to L_{71} and M_{72} to M_{77}

Row IV: (Left to right): H_2 to H_{11} and H_{12} to H_{26}.

3. MAIN RESULT

3.1 Lattice theoretic properties in the subgroup lattices of the symmetric group s_n

S_n can be considered as the group of all permutations on n symbols1, 2, …... n. Then we may think of S_{n-1} as the subgroup of S_n where S_{n-1} is the collection of all permutations in S_n each of which leave one particular symbol fixed.

There are n such subgroups S_{n-1} of S_n where one subgroup S_{n-1} corresponds to fixation of one symbol among 1, 2, ………, n.

Similarly, by fixing two particular symbols at a time we get nC_2 intervals of the form [{i}, S_{n-2}] in $L(S_n)$ and so on and finally fixing n-1 particular symbols at a time we have n intervals of the form [{i}, S_2] in $L(S_n)$.

Therefore, the study of many of the lattice theoretical properties of $L(S_n)$ reduces to a study on the lattices $L(S_i)$, where $i < n$.

Lemma 3.1.1

$L(S_n)$ is distributive if and only if $n \leq 2$

Proof:

Let $L(S_n)$ be distributive

Suppose that $n > 2$. We know that a lattice is distributive if and only if it contains no sublattice isomorphic to N_5 or M_3. $L(S_n)$ contains $L(S_3)$ as an interval which is not distributive, as $L(S_3)$ is isomorphic to M_4. Hence, $L(S_n)$ is distributive implies that $n \leq 2$.

Conversely, assume that $n \leq 2$, Since $L(S_2)$ is isomorphic to the two element chain it is distributive. Hence the proof.

Lemma: 3.1.2

$L(S_n)$ satisfies the general disjointness condition if and only if $n \leq 3$.

Proof:

Let us assume that $L(S_n)$ satisfies the general disjointness condition

Suppose that $n > 3$, then $L(S_n)$ contains $L(S_4)$ as an interval which does not satisfy the general disjointness condition, since there are K_2, K_3 and L_{11} in $L(S_4)$ such that $K_2 \wedge K_3 = \{i\}$ and $(K_2 \vee K_3) \wedge L_{11} = \{i\}$,

But, $K_2 \wedge (K_3 \vee L_{11}) = K_2 \wedge A_4 = K_2 \neq \{i\}$

Therefore $L(S_n)$ satisfies the general disjointness condition implies that $n \leq 3$

Conversely, assume that $n \leq 3$

Since $L(S_2)$ is isomorphic to B_1 and $L(S_3)$ is isomorphic to M_4 , they satisfy the general disjoint condition . Hence the proof.

Lemma: 3.1.3

In $L(S_4)$, every atom is non modular.

Proof:

Consider an atom among the atoms K_2, K_3, and K_4, say K_2

We have $K_5 \subseteq N_{24}$

Now, $(K_5 \vee K_2) \wedge N_{24} = S_4 \wedge N_{24} = N_{24}$ whereas $K_5 \vee (K_2 \wedge N_{24}) = K_5 \vee \{i\} = K_5$

Hence, $(K_5 \vee K_2) \wedge N_{24} \neq K_5 \vee (K_2 \wedge N_{24})$

Therefore, K_2 is not modular in $L(S_4)$

Similarly, we can prove that K_3 and K_4 are not modular.

By similar argument we can prove that all the other atoms in $L(S_4)$ are not modular. Hence there is no atom in $L(S_4)$ which is modular.

Lemma: 3.1.4

In $L(S_5)$, every atom is non modular.

Proof:

Consider an atom among the atoms M_{72}, M_{73}, …….. M_{77} say M_{72}

We have $L_{37} \subseteq Q_{108}$

Now, $(L_{37} \vee M_{72}) \wedge Q_{108} = S_5 \wedge Q_{108} = Q_{108}$

whereas $L_{37} \vee (M_{72} \wedge Q_{108}) = L_{37} \vee e = L_{37}$

Since $L_{37} \neq Q_{108}$, we have M_{72} is not modular

Similar argument holds for M_{73}, M_{74},…….., M_{77}

Similarly we can prove that all the atoms in $L(S_5)$ are non modular.

Hence there is no atom in $L(S_5)$ which is modular.

Lemma 3.1.5

$L(S_n)$ is supersolvable when n = 2 and n = 3 and $L(S_n)$ is not supersolvable when n = 4 and n = 5

Proof:

We know that any modular lattice is supersolvable

Since $L(S_n)$ is modular when n = 2 and n = 3, it is supersolvable.

When n = 4,

By lemma 3.1.3 we have, no atom in $L(S_4)$ is modular, so there is no maximal chain in $L(S_4)$ with modular elements. Hence $L(S_4)$ is not supersolvable.

When n = 5

By lemma 3.1.4 we have, no atom in $L(S_5)$ is modular, so there is no maximal chain in $L(S_5)$ with modular elements. Hence $L(S_5)$ is not supersolvable.

Note:

For $n > 5$, we cannot decide the supersolvability and it remains an open problem.

Lemma 3.1.6

$L(S_n)$ is Supermodular if and only if $n \leq 3$

Proof:

Let $L(S_n)$ be supermodular

Suppose that $n > 3$, then $L(S_n)$ contains $L(S_4)$ as an interval which is not supermodular. Therefore, the supermodularity of $L(S_n)$ implies that $n \leq 3$.

Conversely, assume that $n \leq 3$ is true.

Since we know that $L(S_2)$ is isomorphic to the two element chain it is distributive and we have every distributive lattice is supermodular. Therefore $L(S_2)$ is supermodular. Similarly $L(S_3)$ is isomorphic to M_4 is supermodular. Hence the proof.

Lemma 3.1.7

$L(S_n)$ is semi-supermodular if and only if $n \leq 3$

Proof:

Let $L(S_n)$ be semi-supermodular

Suppose that $n > 3$, then $L(S_n)$ contains $L(S_4)$ as an interval which is not semi supermodular. For, M_{15}, M_{19}, L_{11}, K_5, $N_{25} \in L(S_4)$

$(M_{15} \vee M_{19}) \wedge (M_{15} \vee L_{11}) \wedge (M_{15} \vee K_5) \wedge (M_{15} \vee N_{25}) = P_{26}$

But, $M_{15} \vee [M_{19} \wedge L_{11} \wedge (M_{15} \vee K_5) \wedge (M_{15} \vee N_{25})] \vee [M_{19} \wedge K_5 \wedge (M_{15} \vee L_{11}) \wedge (M_{15} \vee N_{25})] \vee [M_{19} \wedge N_{25} \wedge (M_{15} \vee L_{11}) \wedge (M_{15} \vee K_5)] \vee [L_{11} \wedge K_5 \wedge (M_{15} \vee M_{19}) \wedge (M_{15} \vee N_{25})] \vee [L_{11} \wedge N_{25} \wedge (M_{15} \vee M_{19}) \wedge (M_{15} \vee K_5)] \vee [K_5 \wedge N_{25} \wedge (M_{15} \vee M_{19}) \wedge (M_{15} \vee L_{11})] = M_{15} \neq P_{26}$

Therefore $L(S_4)$ is not semi-supermodular

Therefore, the semi-supermodularity of $L(S_n)$ implies that $n \leq 3$.

Conversely, assume that $n \leq 3$ is true.

Since we know that $L(S_2)$ is isomorphic to the two element chain which is distributive and therefore supermodular. Also $L(S_3)$ is isomorphic to M_4 which is supermodular.

We know that any supermodular lattice is semi-supermodular

By lemma 3.1.6, $L(S_n)$ is semi-supermodular if and only if $n \leq 3$.

Lemma 3.1.8

$L(S_n)$ is pseudo complemented if and only if $n \leq 2$

Proof:

Let $L(S_n)$ be pseudo complemented

Suppose that $n > 2$, then $L(S_n)$ contains $L(S_3)$ as an interval which is isomorphic to M_4 , which is not pseudo complemented.

Therefore, the pseudo complementedness of $L(S_n)$ implies that $n \leq 2$.

Conversely, assume that $n \leq 2$ is true.

Since, $L(S_2)$ is isomorphic to the two element chain, it is pseudo complemented. Hence the proof.

4. CONCLUSION

In this paper we have investigated some of the lattice theoretic properties distributivity, general disjointness condition, supersolvability, supermodularity, semi-supermodularity, pseudo complementedness in $L(S_n)$.

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